金融風險管理季刊
民94，第一卷，第二期，1-15

Application of Quantile Regression to Estimation of Value at Risk

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Abstract

This paper compares performances of the 1 % and 5 % Nikkei 225 VaR calculations with the quantile regression approach to those with the conventional variance-covariance approach and draws several conclusions. First, VaR calculations with the quantile regression approach outperform those with the variance-covariance approach. Second, advantages of the quantile regression approach are more obvious in calculating VaRs for longer holding periods. Third, the quantile regression approach combined with t-GARCH(1, 1) one-step-ahead volatility forecasts provides the best estimates for 1 % and 5 % VaRs of Nikkei 225. Finally, calculations with the variance-covariance approach under distribution assumption are not recommended for estimation of VaR for the log returns of Nikkei 225.

Keywords: VaR, quantile regression, volatility, GARCH
JEL Classification: C12, C22

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1. Introduction

Value at Risk (VaR) has been an extensively used technique for measuring market risk of a portfolio. VaR is simply an estimate of a specific percentile for the distribution of a certain portfolio's market value change over a given holding period. In practice, Bankers Trust reports its daily 1% VaRs and J.P. Morgan reports 5 % ones. More comprehensive discussions are available in Duffie and Pan (1997) and Jorion (1997).

Conventionally, the variance-covariance approach with an assumption of conditional "lognormal" returns is used to calculate a portfolio's VaRs. However, log returns are frequently found not normally distributed, see Boudoukh et al. (1997) and Hull and White (1998). This motivates the employment of a distribution-free approach to estimate the distribution of returns. In literature, the quantile regression approach suggested by Koenker and Basset (1982) provides an estimation for a specific quantile under a conditional distribution. Engle and Manganelli (1999) formulate VaRs based on a conditional autoregressive Value at Risk model (CAViaR) and estimate them through nonlinear quantile regression. Taylor (1999) presents a procedure to estimate a conditional quantile model which is employed to calculate VaRs for a portfolio over holding period k. This new method is found comparable to conventional methods in forecasting the 1 % VaRs for several foreign exchange rates. To allow nonlinearity for the forecasting model, Taylor (2000) applies the quantile regression neural network method introduced by White (1992) to estimation of conditional quantiles.

In Taylor (1999, 2000), determination of the quantile regression model is based on an inspection of bootstrapped coefficient standard errors and a pseudo $R2$ statistic. Since financial data feature leptokurtosis, heavy tail and autocorrelation, the conventional bootstrapping methods are inappropriate for construction of coefficient confidence intervals, see Chen and Chen (2001). Based on the results of simulation studies, Chen and Chen (2001) demonstrate that rank-inverse test suggested by Koenker and Machado (1999) is applicable for regression models with GARCH (generalized autoregressive conditional heteroskedasticity) errors. The rank-inverse test therefore is employed for detecting the significance of coefficient in this paper. In addition to Gaussian GARCH used by Taylor (1999, 2000), several other forecasts of volatility including EWMA and t-GARCH methods, are considered as well in the quantile regression models.

The rest of this paper is organized as follows. Section 2 outlines construction of the forecasting model of VaRs for a portfolio.

Several other forecasts of multiperiod volatility are also studied in section 2. Section 3 includes basic introduction to the quantile regression and construction of the distribution of multiperiod log returns using quantile regression approach. Comparisons among various forecasting models for multiperiod VaRs of Nikkei 225 are investigated in section 4. Conclusions and suggestions are presented in section 5.

2. Variance-Covariance Approach for Calculation of VaR

Formally, a VaR calculation aims at making a statement that "We are (100-τ) % certain that we shall not lose more than *V* dollars in the next *k* days", where *V* is the VaR, (100-τ) % is the confidence level, and *k* is the time horizon. Therefore, VaR is an estimate of the τth percentile of probability distribution of the market value change for a portfolio. Suppose the log returns of a portfolio for holding period *k* at time *t* is denoted as $\gamma_{t,k}=\ln(P_{t+k})-\ln(P_t)$ with a density function $f(\gamma_{t,k})$, where P_t is the market value of the portfolio at time *t*. Given the density, the τ% value at risk ($V_{t,k}(\tau)$) of this portfolio for holding period *k* is determined as

$$\int_{-\infty}^{V_{t,k}(\tau)} f(r_{t,k})dr_{t,k} = \tau\ \%.$$

Apparently, value at risk is a certain percentile of the distribution for *k*-period returns. Therefore, knowledge about the density function of the portfolio returns is crucial for calculation of VaRs.

2.1 Calculation of VaRs under Normal Distribution

Under assumption of normal distribution for log returns, the τth VaR of a portfolio for holding period *k* can be calculated with the J.P. Morgan RiskMetrics variance-covariance approach as

$$V_{t,k}(\tau) = \mu_{t,k} + Z_\tau \sigma_{t,k}.$$

where Z_τ is the τth percentile of a standard normal distribution and $\mu_{t,k}$ and $\sigma^2_{t,k}$ are mean and variance of $\gamma_{t,k}$, respectively. VaR at time *t* is usually determined with the forecasts of $\mu_{t,k}$ and $\sigma_{t,k}$. In literature, $\mu_{t,k}=0$ is assumed under the efficient market hypothesis. This procedure has been discussed comprehensively by Kroner et al.(1995) and Alexander and Leigh (1997). As to the calculations of VaR under normal distribution are discussed as follows.

2.1.1 Exponential Weighted Moving Average (EWMA) Method

J.P. Morgan RiskMetrics™ uses EWMA method to forecast one-step-ahead volatility log returns. Denote historical one-period log returns of a portfolio $\gamma_t=\ln(P_t)-\ln(P_{t-1})$.

The m-period EWMA estimator for the volatility of one-period log returns is defined as

$$\hat{\sigma}_{t+1} = \frac{r_t^2 + \lambda r_{t-1}^2 + \lambda^2 r_{t-2}^2 + \cdots + \lambda^m r_{t-m}^2}{1 + \lambda + \lambda^2 + \cdots + \lambda^m}, \quad (1)$$

where $0 < \lambda < 1$. The denominator converges to $1/(1-\lambda)$ as $m \to \infty$, an infinite EWMA therefore can be written as

$$\begin{aligned} \hat{\sigma}_{t+1}^2 &= (1-\lambda)\sum_{i=0}^{\infty} \lambda^{i-1} r_{t-i}^2 \\ &= (1-\lambda) r_t^2 + \lambda \hat{\sigma}_t^2. \end{aligned} \quad (2)$$

J.P. Morgan RiskMetrics™ sets $m = 250$ and $\lambda = 0.94$ in (1). Under the efficient market hypothesis, $\mu_{t,k} = 0$, the τth VaR estimated with the variance-covariance approach is

$$V_{t,k}(\tau) = Z_\tau \sqrt{k}\, \hat{\sigma}_{t+1}, \quad (3)$$

in which $\hat{\sigma}_{t,k} = \sqrt{k}\, \hat{\sigma}_{t+1}$. This method is denoted as VaR.EWMA.

2.1.2 Gaussian GARCH(1,1) Method

Another popular method for predicting σ_{t+1} is the GARCH(1,1) method. The one-period log return, r_t, following a Gaussian GARCH(1, 1) model is written as

$$\begin{aligned} & r_t = \mu + v_t, \\ & v_t \mid \Phi_{t-1} = \sigma_t \epsilon_t, \ \epsilon_t \sim N(0,1), \\ & \sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \end{aligned}$$

where Φ_{t-1} denotes the information up to time t - 1. The one-step-ahead GARCH(1,1) conditional variance is given by

$$\sigma_{t+1}^2 = \omega + \alpha \epsilon_t^2 + \beta \sigma_t^2.$$

And, the s-step-ahead forecast is given by the recursive expression for $s > 1$,

$$\sigma_{t+s}^2 = \omega + (\alpha + \beta)\sigma_{t+s-1}^2.$$

The forecast of k-period GARCH(1,1) conditional variance therefore is

$$\sigma_{t,k}^2 = \sum_{i=1}^{k} \sigma_{t+i}^2 = \frac{\omega k}{1-\alpha-\beta} + \left[\sigma_{t+1}^2 - \frac{\omega}{1-\alpha-\beta}\right] \left[\frac{1-(\alpha+\beta)^k}{1-\alpha-\beta}\right]. \quad (4)$$

Given the estimated parameters in the GARCH model (i.e., $\hat{\mu}$, $\hat{\omega}$, $\hat{\alpha}$, and $\hat{\beta}$), the τth quantile of VaR is calculated as

$$V_{t,k}(\tau) = \hat{\mu} + Z_\tau \left[\frac{\hat{\omega} k}{1-\hat{\alpha}-\hat{\beta}} + \left[\hat{\sigma}_{t+1}^2 - \frac{\hat{\omega}}{1-\hat{\alpha}-\hat{\beta}}\right] \left[\frac{1-(\hat{\alpha}+\hat{\beta})^k}{1-\hat{\alpha}-\hat{\beta}}\right]\right]^{1/2}, \quad (5)$$

where

$$\hat{\sigma}_{t+1}^2 = \hat{\omega} + \hat{\alpha} e_t^2 + \hat{\beta}\hat{\sigma}_t^2,$$

e_t is the fitted residual. This method is denoted as VaR.GARCH.

2.2 Calculation of VaRs under Non-Gaussian Distribution

As Z_τ is used, calculations of VaR mentioned above are under normal assumption on the one-period log returns. However, as is well known that financial time series are not normally distributed in reality, Z_τ is inappropriate to calculate VaRs. Two alternatives are considered in this paper to obtain VaRs for a portfolio not normally distributed.

2.2.1 *t*-GARCH(1,1) Method

To avoid imposing normal distribution assumption, Bollerslev (1986) suggests the *t*-GARCH models to formulate the conditional heteroskedastic variances of r_t. That is,

$$\begin{aligned} r_t &= \mu + v_t, \qquad v_t|\Phi_{t-1} = \sigma_t \epsilon_t, \\ \epsilon_t &\sim \frac{\nu^{-1/2}}{Beta(1/2, \nu/2)} \left(1 + \frac{\epsilon_t^2}{\nu}\right)^{-(\nu+1)/2}, \\ \sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2, \end{aligned}$$

where ν is a measure of platykurtosis, i.e., "fatness" of the tails for the distribution of ϵ_t and $Beta(\cdot\cdot\cdot)$ denotes a Beta function. After the parameters in *t*-GARCH model (i.e., $\hat{\mu}$, $\hat{\omega}$, $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\nu}$) are estimated, the τth percentile of VaR is then calculated as

$$V_{t,k}(\tau) = \hat{\mu} + t_\tau(\hat{\nu}) \left[\frac{\hat{\omega} k}{1-\hat{\alpha}-\hat{\beta}} + \left[\hat{\sigma}_{t+1}^2 - \frac{\hat{\omega}}{1-\hat{\alpha}-\hat{\beta}}\right] \left[\frac{1-(\hat{\alpha}+\hat{\beta})^k}{1-\hat{\alpha}-\hat{\beta}}\right]\right]^{1/2}, \qquad (6)$$

where

$$\hat{\sigma}_{t+1}^2 = \hat{\omega} + \hat{\alpha}\, e_t^2 + \hat{\beta}\, \hat{\sigma}_t^2,$$

e_t is the fitted residual. This method is denoted as VaR.tGARCH.

2.2.2 Calculation of VaRs without Distribution Assumption

After the Taylor expansion on $(\alpha+\beta)^k$ is conducted, Taylor (1999) points out (4) can be rewritten as

$$\begin{aligned} \sigma_{t,k}^2 &= a_0 + a_1 k + a_2 k \sigma_{t+1}^2 + a_3 \sigma_{t+1}^2 + \cdots \\ &=: g(\sigma_{t+1}^2, k). \end{aligned} \qquad (7)$$

The distribution function of returns for a portfolio over holding period k can be constructed as

$$V_{t,k} = F(\mu, g(\sigma_{t+1}^2, k)). \qquad (8)$$

Under the efficient market hypothesis, $\mu = 0$, (8) becomes

$$V_{t,k} = F(g(\sigma_{t+1}^2, k)) = G(\sigma_{t+1}^2, k). \qquad (9)$$

As $k^{1/2}\sigma_{t+1}$ appears in (3) and $k, k\sigma_{t+1}^2$ and σ_{t+1}^2 appear in (4), they are candidates of the terms in the function $g(\cdot)$ in (7). Since data may not be generated as a GARCH(1,1) process, some other functions of k and σ_{t+1}, say, $k\sigma_{t+1}$ and $k^2\sigma^2_{t+1}$, can also be considered in $g(\cdot)$.

Under assumption of linearly functional form for G(‧，‧), Taylor (1999) determines the τth VaR by estimating the following quantile regression model:

$$\begin{aligned} V_{t,k}(\tau) &= \beta_{1,\tau} + \beta_{2,\tau} k^{1/2} + \beta_{3,\tau} k \\ &\quad + \beta_{4,\tau} k^2 + \beta_{5,\tau}\sigma_{t+1} + \beta_{6,\tau}\sigma^2_{t+1} \\ &\quad + \beta_{7,\tau} k^{1/2}\sigma_{t+1} + \cdots + u_{t,\tau}. \end{aligned} \tag{10}$$

To relax the linear assumption on G(σ^2_{t+1}, k), Taylor (2000) estimates conditional quantile using quantile regression neural network method introduced by White (1992) instead.The linear function of G(σ^2_{t+1}, k), however, will be retained in this paper.

3. Estimation of VaRs with Quantile Regressions

The *linear*[1] quantile regression models developed by Koenker and Bassett (1978) are briefly introduced bellow. Suppose the relationship between response variable y_t and explanatory variable vector x_t (p×1 vector) at τth quantile is specified as

$$Q_\tau(y_t|x_t) = x_t^{'}\beta_\tau.$$

It is ready to see

$$\tau = \int_{-\infty}^{x_t^{'}\beta_\tau} f_y(z|x_t)dz$$

where $f_y(\cdot|x_t)$ is the density function of Y conditional on x_t. A quantile regression model can be writen as

$$y_t = x_t^{'}\beta_\tau + u_{t,\tau}$$

where x_{t1} =1,and $\{u_{t,\tau}\}$ are errors with distribution function F. $Q_\tau(u_{t,\tau}|x_t) = 0$ immediately follows the above definitions. In the light of fashion for sample quantile estimators, the point estimators $\hat{\beta}_\tau$ of the linear quantile regression parameters β_τ are obtained as

$$\begin{aligned} \hat{\beta}_\tau &:= \arg\min_{\beta\in\mathbf{R}^p} \sum_{t=1}^{T} \rho_\tau(y_t - x_t^{'}\beta) \\ &= \arg\min_{\beta\in\mathbf{R}^p} \frac{1}{T}\Bigg[\sum_{t\in\{t:y_t\ge x_t^{'}\beta\}} \tau|y_t - x_t^{'}\beta| + \\ &\qquad \sum_{t\in\{t:y_t< x_t^{'}\beta\}} (1-\tau)|y_t - x_t^{'}\beta| \Bigg], \end{aligned}$$

where ρ_τ (‧) is usually called the “check fucntion”.

[1] Jurečková and Procházka (1993) have proved the consistency and asymptotic normality of the nonlinear quantile regression estimators. Afterwards, related applications and theoretical studies are also extensively discussed in literature, e.g. Koenker, et al. (1997), Engle and Manganelli (1999) and Taylor (2000), etc. However, the discussions are confined to linear quantile regression methods in this paper.

The notoriously time-consuming computation for quantile regression estimation has been reduced considerably with the technique of linear programming suggested by Koenker and d'Orey (1987) and the modern high-speed computer nowadays. In addition, an interior point method for linear programming proposed by Koenker and Park (1996) and Koenker et al. (1997) has been shown comparable to least squares in computation. With stochastic equicontinuity, Fitzenberger (1997) derives the asymptotic normality of quantile regression estimators for models with strong mixing variables. With the well established asymptotic normality, conventional goodness-of-fit and significance tests are readily applicable.

A linear programming problem can be analyzed in two ways. The original problem is conventionally called *primal* problem and the associated problem *dual*. Based on the estimations from the primal problem, Koenker and Machado (1999) suggest a so-called pseudo-R^2 criterion for model selection. Suppose the linear quantile regression model is rewritten as

$$y_t = x_{1t}'\beta_{1\tau} + x_{2t}'\beta_{2\tau} + u_{t,\tau}, \qquad (11)$$

and denote $\hat{\beta}_\tau$ as the minimizer of the unrestricted objective function

$$\hat{V}(\tau) = \arg\min_{\beta\in\mathbb{R}^p} \sum_{t=1}^{T} \rho_\tau(y_t - x_t'\beta).$$

Additionally,inder the q-dimensional linear restriction that HO: $\beta_{2\tau}=0$,denpte $\tilde{\beta}_\tau = (\tilde{\beta}_{1\tau}', 0')'$ as the minimizer of the restricted criterion function

$$\tilde{V}(\tau) = \arg\min_{\beta\in\mathbb{R}^{p-q}} \sum_{t=1}^{T} \rho_\tau(y_t - x_{t1}'\beta).$$

Thereupon, we can define the goodness-of-fit criterion which is analogous to R^2 in least squares regression as

$$R^1(\tau) = 1 - \frac{\hat{V}(\tau)}{\tilde{V}(\tau)}.$$

$R^1(\tau)$ measures the relative success of the quantile regression model at the quantile of interest in terms of a weighted sum of absolute residuals. Like R^2, $0\leq R^1(\tau)\leq 1$.Unlike R^2 measuring a global goodness of fit over the whole conditional distribution, $R^1(\tau)$ measures the local goodness of fit for a certain quantile. $R^1(\tau)$ is therefore able to explore more information at different portions of the conditional distribution. In addition, $R^1(\tau)$ process can be used to construct test statistics for joint hypotheses. Hence it will be employed for model selection. In contrast to $R^1(\tau)$ derived from the results of primal solution, the rank-inverse test originated from the rank test of Gutenbrunner et al. (1993) and constructed by Koenker and Machado (1999) is from the results of dual solutions. As financial time series feature leptokurtosis, heavy tail and autocorrelation, the rank-inverse test is employed to test the significance of each regression coefficient because of its excellent performance demonstrated by Chen and Chen (2001).

3.1 Rank-inverse Test in Linear Quantile Regression Model

The duality of ranks and quantiles is well known in statistics. Gutenbrunner and Jurečková (1993) showed that the solutions of dual problem for linear program which is formulated for computing regression quantiles generalize the duality of ranks and quantiles to linear regression models. The dual solution called regression rank-score process establishes the link between linear rank statistics and regression quantiles. Integrating the regression rank-scores with respect to an appropriate signed measure on (0,1), one can use it to construct tests[2]. The rank-inverse test is constructed by integrating the score generating function with respect to regression rank-scores. The rank-inverse test designed for testing quantile regression estimators circumvents the difficult problem of estimating sparsity function in time series. The testing procedures for rank-inverse test are specified as follows.

Consider a linear model $y = X_1\beta_1 + X_2\beta_2 + u$ and the null hypothesis $H_0 : \beta_2 = \xi \in \mathbf{R}^q$ is undertaking given the significant level, α . The regression rank-scores, $\hat{a}_T$, Compute are computed by solving

$$\arg\max\{(y - X_2\xi)' a | X_1' a = (1-\tau) X_1' \ell, \quad a \in [0,1]^T\}$$

where ℓ is a column vector of ones. For the sake of computation, the τ-quantile score function is considered

$$\varphi_\tau(\omega) = \tau - I(\omega < \tau)$$

and

$$\hat{b}_{Tt}(\xi) = -\int_0^1 \varphi_\tau(\omega) d\hat{a}_{Tt}(\omega) = \hat{a}_{Tt}(\tau) - (1-\tau)$$

with

$$\bar{\varphi} = \int_0^1 \varphi_\tau(\omega) d\omega = 0$$

[2] Gutenbrunner and Jurečková(1993) discussed three broad classes of statistics which have various applications.

and[3]

$$A^2(\varphi_\tau) = \int_0^1 (\varphi_\tau(\omega) - \bar{\varphi})^2 d\omega \\ = \tau(1-\tau).$$

Form the q-vector $S_T(\xi) = T^{-1/2} X_2' \hat{b}_T(\xi)$, where $\hat{b}_T(\xi)$ is the vector with elements $\hat{b}_{Tt}(\xi)$. [4]Under the null and some suitable conditions, Koenker and Machado (1999) show that

$$S_T \xrightarrow{d} N(0, A^2(\varphi_\tau) Q_T)$$

where $Q_T = T^{-1} X_2' (I - X_1 (X_1' X_1)^{-1} X_1') X_2$ and then, the test statistic

$$TR_T(\tau) = \frac{S_T'(\xi) Q_T^{-1} S_T(\xi)}{A^2(\varphi_\tau)} \xrightarrow{d} \chi_q^2. \quad (12)$$

Based on the results of simulation studies, Chen and Chen (2001) demonstrate that rank-inverse test suggested by Koenker and Machado (1999) is applicable for regression models with GARCH (generalized autoregressive conditional heteroskedasticity) errors. The rank-inverse test therefore is employed for detecting the significance of coefficient in this paper.

3.2 Construction of Quantile Regression Models for Estimation of VaR

In this paper, the empirical study aims at a single asset, Nikkei 225 stock index. There are 1000 observations of Nikkei 225 from 5/1/1996 till 5/19/2000 denoted as p_t obtained from Taiwan Economic Journal Data Bank. The k-period rates of log returns, $r_{k,t}$, are computed with $r_{k,t} = \ln(p_t) - \ln(p_{t-k})$. Returns for holding periods of 1, 3, 5, 7, 10, 12 and 15 days are considered. As Duffie and Pan (1997) report the holding period of two weeks has been adopted by various organizations as a standard for VaR calculations, it is reasonable to stop estimation at a holding period of 15 days. The variable of holding

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$$\begin{aligned} \bar{\varphi} &= \int_0^1 \varphi_\tau(\omega) d\omega = \int_0^1 [\tau - I(\omega < \tau)] d\omega \\ &= \int_0^\tau [\tau - I(\omega < \tau)] d\omega + \int_\tau^1 [\tau - I(\omega < \tau)] d\omega \\ &= (\tau\omega - \tau)|_0^\tau + \tau\omega|_\tau^1 = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} A^2(\varphi_\tau) &= \int_0^1 (\varphi_\tau(\omega) - \bar{\varphi})^2 d\omega = \int_0^1 [\tau - I(\omega - \tau)]^2 d\omega \\ &= \int_0^1 [\tau^2 - 2\tau I(\omega < \tau) + I^2(\omega < \tau)] d\omega = \tau(1-\tau). \end{aligned}$$

4 Gutenbrunner and Jurečková(1993) showed that the regression rank-score process, $\{T^{-1/2} d_T \hat{b}_T(\tau)\}$, converges weakly to the Brownian bridge in the uniform topology on C[0,1] provided that d_T is a suitably normalized triangular array of constants.

period is defined as follows.

$$\boldsymbol{k} = [1 \times \boldsymbol{\ell}', 3 \times \boldsymbol{\ell}', 5 \times \boldsymbol{\ell}', 7 \times \boldsymbol{\ell}', 10 \times \boldsymbol{\ell}', 12 \times \boldsymbol{\ell}', 15 \times \boldsymbol{\ell}']',$$

where $\boldsymbol{\ell}$ is a 1000 × 1 vector with each element of 1. In addition, the multiperiod return variable is denoted as

$$\boldsymbol{r} = [\boldsymbol{r}_1', \boldsymbol{r}_3', \boldsymbol{r}_5', \boldsymbol{r}_7', \boldsymbol{r}_{10}', \boldsymbol{r}_{12}', \boldsymbol{r}_{15}']',$$

where γ_k is a 1000 × 1 column vector with elements $\gamma_{k,t}$. For each γ_k , either ARMAGARCH or ARMA-*t* GARCH model has been fitted and no ARMA coefficient is found significant. The efficient market hypothesis therefore is not rejected.

The volatility variable is denoted as

$$\hat{\boldsymbol{\sigma}} = \left[[\hat{\sigma}_{t+1}]', [\hat{\sigma}_{t+1}]', [\hat{\sigma}_{t+1}]', [\hat{\sigma}_{t+1}]', [\hat{\sigma}_{t+1}]', [\hat{\sigma}_{t+1}]', [\hat{\sigma}_{t+1}]' \right]',$$

where $[\hat{\sigma}_{t+1}]$ is a 1000 × 1 vector containing one-step-ahead volatility forecasts, $\hat{\sigma}_{t+1}$, which are estimated with either Gaussian or *t*-GARCH(1, 1) model using γ_1. $\hat{\sigma}_{t+1}$ forecasted with estimated GARCH(1, 1) model is as follows

$$\hat{\sigma}_{t+1}^2 = 0.0706 + 0.0833\, e_t^2 + 0.8875\, \hat{\sigma}_t^2, \quad (13)$$

where $e_t = r_t + 0.0025$. And $\hat{\sigma}_{t+1}$ forecasted with estimated *t*-GARCH(1, 1) model is as follows

$$\hat{\sigma}_{t+1}^2 = 0.0783 + 0.0888\, e_t^2 + 0.8784\, \hat{\sigma}_t^2, \quad (14)$$

where $e_t = \gamma_t + 0.0035$ and $\hat{\nu} = 7.3923$.

In this paper, the linear quantile regression models of (10) are constructed with dependent variable γ and explanatory variables including

$$\mathbf{1},\ \boldsymbol{k}^{1/2},\ \boldsymbol{k},\ \hat{\boldsymbol{\sigma}},\ \hat{\boldsymbol{\sigma}}^2,\ \boldsymbol{k}^{1/2} \odot \hat{\boldsymbol{\sigma}},\ \boldsymbol{k}^{1/2} \odot \hat{\boldsymbol{\sigma}}^2,\ \boldsymbol{k} \odot \hat{\boldsymbol{\sigma}},\ \boldsymbol{k} \odot \hat{\boldsymbol{\sigma}}^2,$$

where $\odot$ denotes the *Hadamard product(direct product)*[5] and **1** is defined as

$$\mathbf{1} = [\boldsymbol{\ell}', \boldsymbol{\ell}', \boldsymbol{\ell}', \boldsymbol{\ell}', \boldsymbol{\ell}', \boldsymbol{\ell}', \boldsymbol{\ell}']'.$$

The rank-inverse test is employed for detecting the significance of each coefficient and the goodness-of-fit measure $R^1(\tau)$ for evaluating these quantile regression models. Based on these criteria, estimation of the selected quantile regression model is

5 The Hadamard product or direct product of **A** and **B** is the matrix

$$\hat{V}_{t,k}(1) = 0.705 + 0.315^{*}\,k + 0.011\,k\,\hat{\sigma}_{t+1} - 2.649^{*}\,k^{1/2}\,\hat{\sigma}_{t+1}, \quad (15)$$

for $\tau = 1$ and

$$\hat{V}_{t,k}(5) = 0.135 + 0.198^{*}\,k - 0.046\,k\,\hat{\sigma}_{t+1} - 1.674^{*}\,k^{1/2}\,\hat{\sigma}_{t+1}, \quad (16)$$

for $\tau = 5$, respectively. The asterisk indicates the rank-inverse test statistic is significant at 5 %. Here the forecasts of $\hat{\sigma}_{t+1}$ are estimated with the Gaussin GARCH(1,1) model. This method is denoted as VaR.QR.GARCH. The forecasts of $\hat{\sigma}_{t+1}$ are estimated with *t*-GARCH(1, 1) as well and the estimation of selected quantile regression model is

$$\hat{V}_{t,k}(1) = 0.7054^{*} + 0.3316^{*}\,k + 0.0042\,k\,\hat{\sigma}_{t+1} - 2.6484^{*}\,k^{1/2}\,\hat{\sigma}_{t+1} \quad (17)$$

for $\tau = 1$ and

$$\hat{V}_{t,k}(5) = 0.2501 + 0.1932^{*}\,k - 0.0379\,k\,\hat{\sigma}_{t+1} - 1.7165^{*}\,k^{1/2}\,\hat{\sigma}_{t+1} \quad (18)$$

for $\tau = 5$, respectively. This method is denoted as VaR.QR.tGARCH. To complete the comparisons, $\hat{\sigma}_{t+1}$ estimated with EWMA is also considered in constructing quantile regression models. The estimation of selected quantile regression model is

$$\hat{V}_{t,k}(1) = -0.0173 + 0.1479\,k + 0.0864\,k\,\hat{\sigma}_{t+1} - 2.4217^{*}\,k^{1/2}\,\hat{\sigma}_{t+1} \quad (19)$$

for $\tau = 1$ and

$$\hat{V}_{t,k}(5) = -0.4173 + 0.1294^{*}\,k - 0.0629\,k\,\hat{\sigma}_{t+1} - 1.3694^{*}\,k^{1/2}\,\hat{\sigma}_{t+1} \quad (20)$$

for $\tau = 5$, respectively. This method is denoted as VaR.QR.EWMA.

4. Comparisons among Different VaR Estimations

In the previous sections, three calculations, VaR.EWMA, VaR.GARCH, and VaR.tGARCH, with the variance-covariance approach and another three, VaR.QR.EWMA, VaR.QR.GARCH, and VaR.QR.tGARCH, with the quantile regression approach are illustrated . Empirical comparisons among these six calculations for Nikkei 225 will be conducted in this section.

As the quantile estimates are with forecasted volatilities, their unobservable nature prevents the employment of MSE criterion for comparisons among various VaR estimations. The back-testing criterion instead is used to evaluate performances of these VaR estimations. The most popular back-testing measure for accuracy of quantile estimator is the percentage of returns falling below the quantile estimate $\hat{V}_{t,k}(\tau)$, denoted as $\hat{B}_\tau$. For an accurate estimator of τth quantile, $\hat{B}_\tau$ will be very close to τ%. This criterion has been used by Granger et al. (1989), Alexander and Leigh (1997), and Taylor (1999, 2000) etc. To determine the significance of departure for $\hat{B}_\tau$ from τ%, the following test statistic is considered:

$$(T \times \hat{B}_\tau - T \times \tau\,\%)/\sqrt{T\tau\,\%(1-\tau\,\%)} \xrightarrow{d} N(0,1),$$

where T is the sample size. Performances on forecasting multiperiod 1 % and 5 % VaRs for Nikkei 225 among the six calculations are reported in Table 1 and Table 2, respectively. The boldfaced figure in each column signifies the closest estimate for a given holding period. The asterisked figures indicate rejection of the equality between $\hat{B}_\tau$ and τ .

For the 1 % VaR, five out of seven boldfaced figures and few asterisked figures are observed in the bottom part of Table 1, which indicates calculations with the quantile regression approach have better performance in accuracy than those with the variance covariance approach do. Among the three calculations with the quantile regression approach, VaR.QR.tGARCH and VaR.QR.GACH have the best and the worst performances, respectively. The former has the most boldfaced figures and no asterisked figure at all, yet the latter has no boldfaced figure and one asterisked figure. Although there are two boldfaced figures present in the top part of Table 1, the performances for calculations with the variance-covariance approach deteriorate as the holding period extends, typically for VaR.GARCH and VaR.tGARCH. These results show calculations with the variance-covariance approach under distribution assumptions perform worst. To summarize, calculations with the quantile regression approach outperform those with the variance-covariance approach, in particular for longer holding periods. Moreover, *t*-GARCH one-step-ahead volatility forecasts combined with quantile regression approach is recommended for calculation of the 1 % VaR for Nikkei 225.

As to the 5 % VaR, calculations with the quantile regression approach again are superior to those with the variance-covariance approach. Although the number of boldfaced figure drops to four, the number of asterisked figure remains two. VaR.QR.GARCH has two boldfaced figures yet two asterisked figures are present. Neither VaR.QR.tGARCH nor VaR. QR. EWMA has a single asterisked figure, which suggests their performances are better than VaR. QR. GARCH's. For calculations with the variance-covariance approach, VaR.EWMA has three boldfaced figures, however, one asterisked figure is found with it. VaR.GARCH and VaR.tGARCH perform worst as all figures are asterisked for holding periods longer than 3 days and not a single boldfaced figure is found. To summarize, calculations with the quantile regression approach outperform those with the variance covariance approach, in particular for longer holding periods. Besides, *t*-GARCH and EWMA one-step-ahead volatility forecasts combined with quantile regression approach are recommended for calculation of the 5 % VaR for Nikkei 225.

Results from Table 1 and 2 confirm the quantile regression approach is a useful tool for calculation of VaRs. Its advantage is obvious for both 1 % and 5 % VaRs, especially where holding periods are longer. As to the effect of forecasts for one-step-ahead volatility, the calculation with quantile regression approach combined with *t*-GARCH(1,1) estimate is recommended for calculation of VaRs for Nikkei 225. Nevertheless, the conventional calculations with the variance-covariance approach under either normal or *t*- distribution are not recommended.

5. Concluding Remarks

Comparisons in accuracy of 1 % and 5 % VaR calculations for Nikkei 225 (5/1/1996 — 5/19/2000) are investigated in this paper. In particular, comparisons between calculations with the quantile regression approach and those with the variance-covariance approach either with or without distribution assumption are

Table 1 Performances for Various Calculations of 1 % VaR

	Holding Period						
	1	3	5	7	10	12	15
VaR.GARCH	2.00*	**0.93**	0.67	0.27*	0.00*	0.00*	0.00*
VaR.tGARCH	**0.40**	0.53	0.13*	0.00*	0.00*	0.00*	0.00*
VaR.EWMA	2.13*	1.86*	0.16	0.80	0.40	0.40	0.40
VaR.QR.GARCH	4.13*	1.33	1.46	1.20	1.20	0.93	1.46
VaR.QR.tGARCH	2.26	1.46	**1.06**	**0.93**	0.93	0.66	**1.06**
VaR.QR.EWMA	2.53	2.13*	1.33	1.33	**1.06**	**1.06**	0.66

Note: * Significant at 5 % level. Boldfaced figures mark the closest estimation for VaR estimation for a given holding period.

Table 2 Performances for Various Calculations of 5 % VaR

	Holding Period						
	1	3	5	7	10	12	15
VaR.GARCH	5.73	3.47*	2.80*	2.13*	1.87*	1.66*	1.60*
VaR.tGARCH	4.27	2.40*	1.20*	1.33*	1.20*	0.66*	0.53*
VaR.EWMA	**5.33**	6.66*	6.40	6.26	**5.60**	5.60	**5.20**
VaR.QR.GARCH	6.80*	**4.80**	**5.06**	4.00	3.06*	4.53	5.73
VaR.QR.tGARCH	6.13	5.46	6.00	4.40	3.73	**5.20**	6.00
VaR.QR.EWMA	6.26	5.33	6.53	**5.06**	4.00	4.26	5.73

Note: * Significant at 5 % level. Boldfaced figures mark the closest estimate for VaR estimation for a given holding period.

emphasized. Forecasts of one-step-ahead volatility are estimated by Gaussian GARCH(1,1), *t*-GARCH(1,1) as well as EWMA in these calculations.

This paper draws several conclusions. First, VaR calculations with quantile regression approach outperform those with variance-covariance approach. Second, advantage of the quantile regression approach in calculating VaRs is more significant for longer holding periods. Third, the quantile regression approach combined with *t*-GARCH(1,1) one-step-ahead volatility forecasts provides the best estimates for 1% and 5% VaRs of Nikkei 225. Finally , calculations with the variance-covariance approach under distribution assumption are not recommended for estimation of VaRs for the log returns of Nikkei 225

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